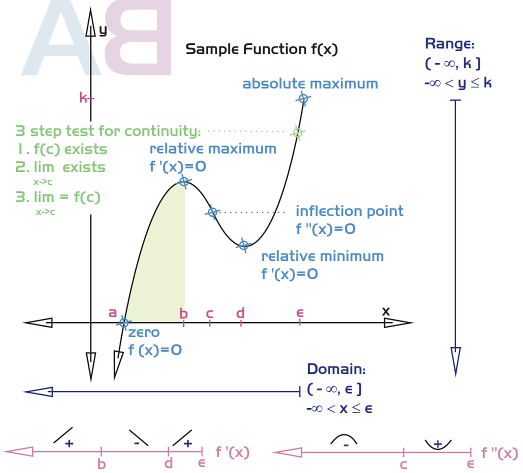
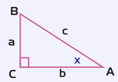
AB|BC calculus

a definitive sheet
by chad valencia, ucla mathematics major
version 2.0.2000, rev 1



Area = $\int_a^b f(x)\,dx$

trig in a nutshell



sin x = a/c = opposite/hypotenuse
cos x = b/c = adjacent/hypotenuse
sec x = c/b = hypotenuse/adjacent
csc x = c/a = hypotenuse/opposite
tan x = a/b = sin x/cos x = opposite/adjacent
cot x = b/a = cos x/sin x = adjacent/opposite

Odd/Even Identities

$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$
$\tan(-x) = -\tan x$
$\cot(-x) = -\cot x$
$\sec(-x) = \sec x$
$\csc(-x) = -\csc x$

Double Angle Identities

$\sin 2x = 2\sin x\cos x$
$\cos 2x = \cos^2 x - \sin^2 x$
$\cos^2 x = \frac{1+\cos 2x}{2}$
$\sin^2 x = \frac{1-\cos 2x}{2}$

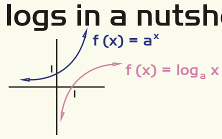
$\sin^2 x + \cos^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$

$\cos(a+b) = \cos a\cos b - \sin a\sin b$
$\sin(a+b) = \sin a\cos b + \cos a\sin b$
$\cos(a-b) = \cos a\cos b + \sin a\sin b$
$\sin(a-b) = \sin a\cos b - \cos a\sin b$

personal notes

logs in a nutshell

$\log x = \log_{10} x$
$\log_{\epsilon} x = \ln x$



$\ln(xy) = \ln x + \ln y$
$\ln(x/y) = \ln x - \ln y$
$\ln x^n = n\ln x$
$\ln \epsilon^x = \epsilon^{\ln x} = x$
$\ln 1 = 0 \quad \ln \epsilon = 1$

if: $a^b = x$
$\log_a x = b$

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \epsilon$

derivatives

definition of the derivative:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

Addition Rule
$f'(u+v) = f'u + f'v$

Product Rule
$f'(uv) = udv + vdu$

Quotient Rule
$f'\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$
(Lo D Hi minus Hi D Lo over Lo Lo)

Power Rule
$f'(x^c) = cx^{c-1}$

Chain Rule
$(f \circ g)' = f'g'$

l'Hôpital's Rule

When $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$ OR $\frac{\infty}{\infty}$

$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$

Mean Value Theorem

$\frac{f(b)-f(a)}{b-a} = f'(c)$

trig derivatives

Standard Trig

$(d/dx)(\csc u) = -\csc u\cot u$
$(d/dx)(\sec u) = \sec u\tan u$
$(d/dx)(\cot u) = -\csc^2 u$
$(d/dx)(\tan u) = \sec^2 u$
$(d/dx)(\cos u) = -\sin u$
$(d/dx)(\sin u) = \cos u$

Inverse Trig

$\frac{d}{dx}\sin^{-1} u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}\tan^{-1} u = \frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}\sec^{-1} u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

transcendental derivatives

$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\epsilon^u = \epsilon^u\frac{du}{dx}$

$\frac{d}{dx}a^u = a^u\frac{du}{dx}\ln a$

volumes & areas

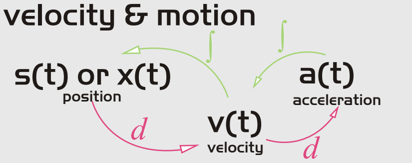
$V_{(sphere)} = \frac{4}{3}\pi r^3$

$SA_{(sphere)} = 4\pi r^2$

$V_{(cone)} = \frac{1}{3}\pi r^2 h$

$A_{\triangle} = \frac{s^2\sqrt{3}}{4}$

velocity & motion



the velocity equation
$s(t) = \frac{1}{2}gt^2 + v_o t + s_o$
$g = -32\ ft/s^2, -9.8\ m/s^2$

integrals

First Fundamental Theorem of Calculus

$\int_a^b f(x)\,dx = F(b) - F(a)$

where F is the antiderivative of f

Second Fundamental Theorem of Calculus (Leibniz's Rule)

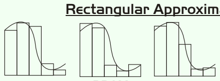
$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v)\frac{dv}{dx} - f(u)\frac{du}{dx}$

Trapezoidal Rule

$T = \frac{b-a}{2n}(y_0 + 2y_1 + 2y_2 + \ldots + 2y_{n-1} + y_n)$
a & b = bounds
n = number of intervals

(Use $h(b_1+b_2)/2$ for trapezoids of different height)

Rectangular Approximation Methods (RAM)



(use b x h for approximation)

Power Rule:

$\int x^a\,dx = \frac{x^{a+1}}{a+1} + C$
$x \neq -1$

Average Value

Avg. $(f(x)) = \frac{1}{b-a}\int_a^b f(x)\,dx$

trigonometric integrals

Standard Trig

$\int \sin x\,dx = -\cos x + C$
$\int \cos x\,dx = \sin x + C$
$\int \sec^2 x\,dx = \tan x + C$
$\int \csc^2 x\,dx = -\cot x + C$
$\int \sec x\tan x\,dx = \sec x + C$
$\int \csc x\cot x\,dx = -\csc x + C$

$\int \tan x\,dx = -\ln|\cos x| + C$
$\int \cot x\,dx = \ln|\sin x| + C$
$\int \sin^2 x\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$
$\int \cos^2 x\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Inverse Trig

$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + C$

$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + C$

$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$

transcendental integrals

$\int \frac{du}{u} = \ln|u| + C$

$\int \ln x\,dx = x\ln x - x + C$

$\int \epsilon^u du = \epsilon^u + C$

$\int a^u du = \frac{a^u}{\ln a} + C$

disc & shell methods

Volume	Disc (no hole)	Disc w/ Hole	Shell
X-Axis	$\pi\int_a^b r^2 dx$	$\pi\int_a^b (R^2 - r^2)dx$	$2\pi\int_c^d r\,h\,dy$
Y-Axis	$\pi\int_c^d r^2 dy$	$\pi\int_c^d (R^2 - r^2)dy$	$2\pi\int_a^b r\,h\,dx$
	r = radius	R = Outside radius, r = inside radius	r = radius, h = height

integration by parts

$$\int u\,dv = uv - \int v\,du$$

Priority:

Logarithmic
Inverse Trig
Algebraic
Trigonometric
Euler's Constant (e)

Tabular Integration

$\int$[(algebraic)(trigonometric/e)]dx
ex: $\int 2x^3\cos x\,dx$

d	Alternating Signs	∫
$2x^3$	+	cosx
$6x^2$	-	sinx
12x	+	-cosx
12	-	-sinx
0		cosx

$2x^3\sin x + 6x^2\cos x - 12x\sin x - 12\cos x + C$

partial fractions

$\frac{px+q}{(x+a)(x+b)} = \frac{A}{(x+a)} + \frac{B}{(x+b)}$

$\frac{px+q}{(x+a)^2} = \frac{A}{(x+a)} + \frac{B}{(x+a)^2}$

$\frac{px^2-qx+r}{(x+a)(x^2+bx+c)} = \frac{A}{(x+a)} + \frac{Bx+C}{(x^2+bx+c)}$

trig substitutions

If you see:	Use:
$a^2 + x^2$	$x = a\tan\theta$
$a^2 - x^2$	$x = a\sin\theta$
$x^2 - a^2$	$x = a\sec\theta$

solved through trig substitution:
$\int \sec u\,du = \ln|\sec u + \tan u| + C$

calculus

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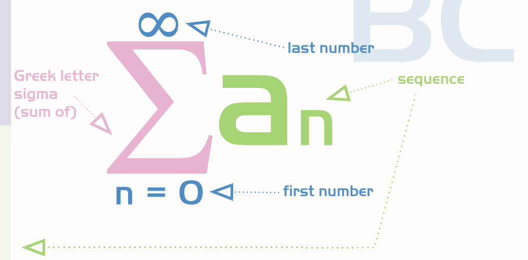


personal notes

improper integrals

P Series Test:

$$\int_1^{\infty} \frac{1}{x^p}$$

Converges if $p > 1$
Diverges if $0 \leq p \leq 1$

Comparison Test

if $f(x) \leq$ convergent function, f(x) is convergent

if $f(x) \geq$ divergent function, f(x) is divergent

Limit Comparison Test:

Let f(x) be a known convergent or divergent function:

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L \quad 0 < L < \infty$$

f(x) & g(x) both converge or both diverge

sequences

Convergence/Divergence:

Let L be a finite number

$$\lim_{n \to \infty} a_n = L$$

Convergent

$$\lim_{n \to \infty} a_n \neq L$$

Divergent

Limits of Common Sequences:

$$\lim_{n \to \infty} \frac{\ln n}{n} = 0$$

$$\lim_{n \to \infty} \frac{x^n}{n!} = 0$$

$$\lim_{n \to \infty} x^n = 0 \quad |x| < 1 \text{ (fraction)}$$

$$\lim_{n \to \infty} \sqrt[n]{n} = \lim_{n \to \infty} n^{\frac{1}{n}} = 1$$

$$\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

series

Geometric Series

Finite Series: $S_n = \frac{a(1-r^n)}{(1-r)}$

Infinite Series: $\sum_{n=1}^{\infty} a r^{n-1} = \frac{a}{1-r}$, $|r| < 1$

if $|r| \geq 1$, series diverges

Nth Term Test for Divergence

Given: $\sum_{n=k}^{\infty} a_n$ If: $\lim_{n \to \infty} a_n \neq 0$ series is divergent

Integral Test

$$\sum_{n=k}^{\infty} a_n \blacktriangleright \int_k^{\infty} a_x \, dx$$

If integral converges, series converges
If integral diverges, series diverges

Comparison Test

if $\Sigma a_n \leq$ convergent series, Σa_n is convergent
if $\Sigma a_n \geq$ divergent series, Σa_n is divergent

Limit Comparison Test

Let Σb_n be a known convergent or divergent series:

$$\lim_{n \to \infty} \frac{\Sigma a_n}{\Sigma b_n} = \rho \quad 0 < \rho < \infty$$

Σa_n & Σb_n both converge or both diverge

Alternating Series Test (AST)

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

Converges if:
1. All terms are positive
2. $a_n > a_{n+1}$ for all n.
3. $\lim_{n \to \infty} a_n = 0$

Ratio Test

$$\rho = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}$$

Root Test

$$\rho = \lim_{n \to \infty} \sqrt[n]{a_n}$$

Converges if $\rho < 1$
No conclusion if $\rho = 1$
Divergent if $\rho > 1$

Power Series

1. Use ratio test on the absolute value of the series.
2. Set $\rho < 1$ to find the interval
3. To check bounds, plug into original equation
4. Take ½ of the interval to find radius of convergence

Error (Alternating Series/Taylor)

Error = Actual - Approximate
Error < First Unused Term

extraneous bc concepts

Arc Length (x-axis):

$$\int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Arc Length (y-axis):

$$\int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy$$

Surface Area:

$$2\pi \int_a^b y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$

Work

Work = Force x Distance
= Density x Volume x Distance
= Density x Sum of Areas x Distance

Hooke's Law for Springs

Force = f(x) = kx
(k is a constant, x is the distance)
Work = ò f(x)dx, from position A to position B

Newton's Law of Cooling

$$T - T_s = (T_o - T_s)e^{-kt}$$

T = temperature of object at a given time
Ts = temperature of surroundings
To = temperature at time zero
t = time

parametric

First Derivative:

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

Second Derivative:

$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Arc Length:

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Surface Area (x-axis):

$$2\pi \int_a^b y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

Surface Area (y-axis):

$$2\pi \int_a^b x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$$

vectors

Notation

v = ai + bj
<a,b>

Unit Tangent and Unit Normal Vectors

$$T(t) = \frac{r'(t)}{||r'(t)||}$$

If $T = u_1 i + u_2 j$
$N = -u_2 i + u_1 j$

Speed

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

Vector Length (magnitude/norm):

$$||v|| = \sqrt{a^2 + b^2}$$

Vector Valued Functions

r(t) = x(t)i + y(t)j
r'(t) = x'(t)i + y'(t)j
$\int r(t)dt = (\int x(t)dt + c)i + (\int y(t)dt + c)j$

Unit Vector:

$$\frac{v}{||v||}$$

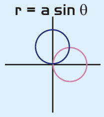
Velocity Equation for Vectors:

$(-\frac{1}{2} g t^2 + S_o)j + V_o t$
where $v_o = \#(\cos t \, i + \sin t \, j)$
= initial velocity/muzzle speed

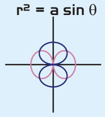
Polar

Basic Shapes
(pink = cosine, blue = sine)

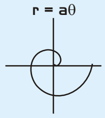
Circles
r = a cos θ
r = a sin θ



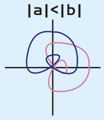
Lemniscates
r² = a cos θ
r² = a sin θ



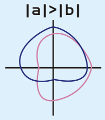
Spiral of Archimedes
r = aθ



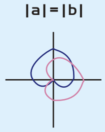
Limacons w/ Inner Loop
r = a + b cos θ
r = a + b sin θ
|a|<|b|



Limacons w/ Dimple
r = a + b cos θ
r = a + b sin θ
|a|>|b|



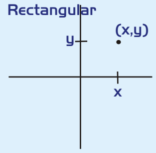
Cardioids
r = a + b cos θ
r = a + b sin θ
|a|=|b|



Roses

r = a cos bθ If b is odd, b = number of petals
r = a sin bθ If b is even, 2b = number of petals

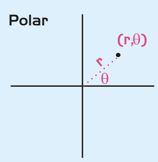
Rectangular



Polar Conversion

(x,y) <-> (r,θ)
x² + y² = r²
x = r cos θ $\tan\theta = \frac{y}{x}$
y = r sin θ

Polar



Polar Slope

$$\frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$$

Polar Area:

$$\frac{1}{2}\int_\alpha^\beta r^2 \, d\theta$$

Polar Surface Area (x-axis):

$$2\pi \int_\alpha^\beta r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Polar Arc Length:

$$\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

Polar Surface Area (y-axis):

$$2\pi \int_\alpha^\beta r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

taylor/maclaurin series

Taylor Polynomial

$$P(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots + \frac{f^n(a)(x-a)^n}{n!}$$

$$\sum_{n=0}^{\infty} \frac{f^n(a)(x-a)^n}{n!}$$

Common MacLaurin Series

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \ldots = \sum_{n=0}^{\infty} (-1)^n x^n$$

hyperbolic trig functions

Every function splits into Even and Odd parts:

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{2}{e^x - e^{-x}}$$

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

sinh 2x = 2 sinh x cosh x
cosh 2x = cosh² x - sinh² x
$\cosh^2 x = \frac{\cosh 2x + 1}{2}$
$\sinh^2 x = \frac{\cosh 2x - 1}{2}$
cosh² x - sinh² x = 1
tanh² x + sech² x = 1
coth² x - csch² x = 1

Standard Hyperbolic Trig Derivatives

(d/dx)(csch u) = - csch u coth u (du/dx)
(d/dx)(sech u) = - sech u tanh u (du/dx)
(d/dx)(coth u) = - csch² u (du/dx)
(d/dx)(tanh u) = sech² u (du/dx)
(d/dx)(cosh u) = sinh u (du/dx)
(d/dx)(sinh u) = cosh u (du/dx)

Inverse Hyperbolic Trig Derivatives

$$\frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \quad u > 1$$

$$\frac{d}{dx}\sinh^{-1} u = \frac{1}{\sqrt{u^2 + 1}} \frac{du}{dx}$$

$$\frac{d}{dx}\tanh^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}, \quad |u| < 1$$

$$\frac{d}{dx}\coth^{-1} u = \frac{1}{1 - u^2} \frac{du}{dx}, \quad |u| > 1$$

$$\frac{d}{dx}\operatorname{sech}^{-1} u = \frac{-1}{u\sqrt{1 - u^2}} \frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d}{dx}\operatorname{csch}^{-1} u = \frac{-1}{|u|\sqrt{u^2 + 1}} \frac{du}{dx}, \quad u \neq 0$$

Hyperbolic Trig Integrals

∫ sinh u du = cosh u + C
∫ cosh u du = sinh u + C
∫ sech² u du = tanh u + C
∫ csch² u du = - coth u + C
∫ sech x tanh u du = - sech u + C
∫ csch x coth u du = - csch u + C